O PROBLEMĂ EXTREMALĂ ÎN CLASA FUNCȚIILOR UNIVALENTE

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ABSTRACT

Fie S clasa funcțiilor $f(z)=z+a_2z^2+\ldots$, $f(0)=0$, $f'(0)=1$ olomorfe şi univalente în discul $|z|<1$.

Pentru $0\le x\le a\le 1$ să considerăm ecuația

$$Re\,[(x^3-a^3)f(x)]=0,\ f\in S. \qquad (1)$$

Notăm cu $\varphi(x)=Re\,[(x^3-a^3)f(x)]$. Deoarece $\varphi(0)=0$ şi $\varphi(a)=0$ rezultă că există $x_0\in(0,a)$ astfel încât: $\varphi'(x_0)=0$.

Scopul acestei lucrări este de a găsi $\max\{x|\ \varphi'(x)=0\}$.

Dacă $\overline{x}$ este $\max\{x|\ \varphi'(x)=0\}$, atunci pentru $x>\overline{x}$ ecuați $\varphi'(x)=0$ nu mai are rădăcini reale. Deoarece S este compactă, există $\overline{x}$.

Această problemă a fost propusă de Petru T. Mocanu în [2]. Vom determina $\overline{x}$ folosind metoda variațională a lui Schiffer-Goluzin [1].

***Cuvinte cheie**: funcție extremală, funcție olomorfă, funcție univalentă*

1. Fie f ϵ S funcția extremală pentru care $\overline{x}$ este atins, cu:

$$Re[3\overline{x}^2 f(\overline{x})+(\overline{x}^3-a^3)f'(\overline{x})]=0. \qquad (1)$$

Acum să considerăm variația funcției f dată de formula Schiffer-Goluzin [1]:

$$f^*(x)=f(x)+\lambda V(x;\zeta;\psi)+0(\lambda^2),\ |\zeta|<1,\ \lambda>0, \qquad (2)$$

ψ număr real, unde:

$$\begin{cases} V(x;\zeta;\psi)=e^{i\psi}\dfrac{f^2(x)}{f(x)-f(\zeta)}-e^{i\psi}\cdot f(x)\cdot[\dfrac{f(\zeta)}{\zeta f'(\zeta)}]^2- \\ -e^{i\psi}\dfrac{xf'(x)}{x-\zeta}\zeta\cdot[\dfrac{f(\zeta)}{\zeta f'(\zeta)}]^2+e^{-i\psi}\dfrac{x^2f'(x)}{1-\overline{\zeta}x}\overline{\zeta}[\dfrac{f(\zeta)}{\zeta f'(\zeta)}]^2 \end{cases} \qquad (3)$$

Să considerăm o variație x^* a lui x:

$$x^*=x+\lambda h+0(\lambda^2),\ h=\frac{\partial x^*}{\partial\lambda}\Big|_{\lambda=0}$$

ce satisface condițiile:

AN EXTREMAL PROBLEM FOR UNIVALENT FUNCTIONS

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ABSTRACT

Let S be the class of functions $f(z)=z+a_2z^2\ldots$, $f(0)=0$, $f'(0)=1$ which are regular and univalent in the unit disk $|z|<1$.

For $0\le x\le a\le 1$ we consider the equation

$$Re\,[(x^3-a^3)f(x)]=0,\ f\in S. \qquad (1)$$

Denote $\varphi(x)=Re\,[(x^3-a^3)f(x)]$. Because $\varphi(0)=0$ and $\varphi(a)=0$ it follows that there is $x_0\in(0,a)$ such that: $\varphi'(x_0)=0$.

The aim of this paper is to find $\max\{x|\ \varphi'(x)=0\}$. If $\overline{x}$ is $\max\{x|\ \varphi'(x)=0\}$, then for $x>\overline{x}$ the equation $\varphi'(x)=0$ does not have real roots. Since S is a compact class, there exists $\overline{x}$.

This problem was first proposed by Petru T. Mocanu in [2]. We will determine $\overline{x}$ by using the variational method of Schiffer-Goluzin [1].

***Key words**: extremal function, regular, univalent*

1. Let f ϵ S be the extremal function for which $\overline{x}$ is attained, which:

$$Re[3x^2f(\overline{x})+(\overline{x}^3-a^3)f'(\overline{x})]=0. \qquad (1)$$

Next we consider a variation of the function f given by Schiffer-Goluzin's formula [1]:

$$f^*(x)=f(x)+\lambda V(x;\zeta;\psi)+0(\lambda^2),\ |\zeta|<1,\ \lambda>0, \qquad (2)$$

ψ real number, where:

$$\begin{cases} V(x;\zeta;\psi)=e^{i\psi}\dfrac{f^2(x)}{f(x)-f(\zeta)}-e^{i\psi}\cdot f(x)\cdot[\dfrac{f(\zeta)}{\zeta f'(\zeta)}]^2- \\ -e^{i\psi}\dfrac{xf'(x)}{x-\zeta}\zeta\cdot[\dfrac{f(\zeta)}{\zeta f'(\zeta)}]^2+e^{-i\psi}\dfrac{x^2f'(x)}{1-\overline{\zeta}x}\overline{\zeta}[\dfrac{f(\zeta)}{\zeta f'(\zeta)}]^2 \end{cases} \qquad (3)$$

Next we consider a variation x^* of x:

$$x^*=x+\lambda h+0(\lambda^2),\ h=\frac{\partial x^*}{\partial\lambda}\Big|_{\lambda=0}$$

which satisfies the conditions:

$|x^*|=x$ şi $\mathrm{Re}[3x^{*2}f^*(x^*)+(x^{*3}-a^3)f'^*(x^*)]=0.$ (4)

Vom nota cu:

$$\begin{cases} f=f(x), w=f(\zeta), \ell=f'(x), m=f''(x), \\ V=V(x;\zeta;\psi), V'=V'_x(x;\zeta;\psi). \end{cases} \tag{5}$$

Folosind (4) şi (5) obținem că funcția extremală w=f(ζ) trebuie să satisfacă următoarea ecuație

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2\left[f+\left(x^3-a^3\right)l\right]-f\left[f+2\left(x^3-a^3\right)l\right]w}{\left(f-w\right)^2}=\frac{\sum_{k=0}^{4} t_k\zeta^k}{\left(x-\zeta\right)^2\left(1-x\zeta\right)^2} \tag{6}$$

unde Re t_0=Re t_4=0, Re t_1=Re t_3 şi t_0, t_1, t_2, t_4, depinzând de x,f,ℓ şi m.

2. Se ştie că funcția extremală w=f(ζ) transformă discul unitate în planul w cu un număr finit de arce analitice. Fie $q=e^{i\theta}$ extremitatea arcului. Polinomul $\sum_{k=0}^{4} t_k\zeta^k$ are rădăcina dublă $\zeta=q$. Rezultă că (6) se mai scrie:

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2\left[f+\left(x^3-a^3\right)l\right]-f\left[f+2\left(x^3-a^3\right)l\right]w}{\left(f-w\right)^2}= \\ =\frac{\left(1-\overline{q}\zeta\right)^2\left[i\left(v-r\right)-2\left(u+iv\right)q\zeta\right]+i\left(v+r\right)q^2\zeta^2}{\left(x-\zeta\right)^2\left(1-x\zeta\right)^2} \tag{7}$$

unde u, v şi r sunt numere reale şi verifică:

$$\begin{cases} (1-2x\cos\theta+x^2\cos 2\theta)[2x(v\sin\theta-u\cos\theta)-x^2(v+r)\sin 2\theta]- \\ -2x\sin\theta(1+x\cos\theta)[v-r-2x(u\sin\theta+v\cos\theta)+x^2(v+r)\cos 2\theta]= \\ =(1-x^2)^2\,\mathrm{Re}\,f, \\ 2x\sin\theta(1+x\cos\theta)[2x(v\sin\theta-u\cos\theta)-x^2(v+r)\sin 2\theta]+ \\ +(1-2x\cos\theta+x^2\cos 2\theta)[v-r-2x(u\sin\theta+v\cos\theta)+x^2(v+r)\cos 2\theta]= \\ =\mathrm{Im}[-(x-a)(1-x^2)^2\ell], \\ \dfrac{5x^3-4ax^2-3x+2a}{(1-x^2)(x-a)}+\dfrac{2x(x-\cos\theta)}{1-2x\cos\theta+x^2}+\dfrac{N_1}{N_2}=\mathrm{Re}\dfrac{xm}{\ell}, \\ -\dfrac{\sin\theta}{1+2x\cos\theta+x^2}+\dfrac{N_3}{N_4}=\mathrm{Im}\dfrac{xm}{\ell} \end{cases} \tag{8}$$

$|x^*|=x$ and $\mathrm{Re}[3x^{*2}f^*(x^*)+(x^{*3}-a^3)f'^*(x^*)]=0.$ (4)

We denote:

$$\begin{cases} f=f(x), w=f(\zeta), \ell=f'(x), m=f''(x), \\ V=V(x;\zeta;\psi), V'=V'_x(x;\zeta;\psi). \end{cases} \tag{5}$$

By using (4) and (5) we obtain that the extremal function w=f(ζ) must satisfy the following equation:

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2\left[f+\left(x^3-a^3\right)l\right]-f\left[f+2\left(x^3-a^3\right)l\right]w}{\left(f-w\right)^2} \tag{6}$$

where Re t_0=Re t_4=0, Re t_1=Re t_3 and t_0, t_1, t_2, t_4, depending of x,f,ℓ, and m.

2. It may be shown that the extremal function w=f(ζ) maps the unit disk onto the entire w – plane slit along a finite number of analytic arcs. Let $q=e^{i\theta}$ be the point which is mapped into an end – point of a slit. The polinomial $\sum_{k=0}^{4} t_k\zeta^k$ has the double root $\zeta=q$. It follows that the equation (6) may be written:

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2\left[f+\left(x^3-a^3\right)l\right]-f\left[f+2\left(x^3-a^3\right)l\right]w}{\left(f-w\right)^2} \\ =\frac{\left(1-\overline{q}\zeta\right)^2\left[i\left(v-r\right)-2\left(u+iv\right)q\zeta\right]+i\left(v+r\right)q^2\zeta^2}{\left(x-\zeta\right)^2\left(1-x\zeta\right)^2} \tag{7}$$

where u, v and r are real numbers and verifies:

unde:

$$N_1 = -2x(v-r)(u\sin\theta + v\cos\theta) + 2[2(u^2+v^2) + (v^2-r^2)\cos 2\theta]x^2 + \\ + 6(v+r)(u\sin\theta - v\cos\theta)x^3 + 2(v+r)^2 x^4;$$

$$N_2 = (v-r)^2 - 4x(v-r)(u\sin\theta + v\cos\theta) + 2[2(u^2+v^2) + (v^2-r^2)\cos 2\theta]x^2 + \\ + 4(v+r)(u\sin\theta - v\cos\theta)x^3 + (v+r)^2 x^4;$$

$$N_3 = 2x(v-r)(-v\sin\theta + u\cos\theta) + 2(v^2-r^2)x^2\sin 2\theta + 2(v+r)\times \\ \times(v\sin\theta + u\cos\theta)x^3 - 4(v+r)(v\sin\theta + u\cos\theta)x^3;$$

$$N_4 = (v-r)^2 - 4x(v-r)(u\sin\theta + v\cos\theta) + 2[2(u^2+v^2) + (v^2-r^2)\cos 2\theta]x^2 \\ + 4(v+r)(u\sin\theta - v\cos\theta)x^3 + (v+r)^2 x^4.$$

Din (7) obţinem:

$$w(\zeta_1) = \frac{f[f+(x-a)\ell]}{f+2(x-a)\ell} \qquad (9)$$

unde $\zeta_1 = \rho\cdot\bar{q}$, $|\zeta_1|<1$, $\rho = \dfrac{u+iv-\sqrt{u^2-r^2+2uv\cdot i}}{i(v+r)}$ (ζ_1 este rădăcina ecuaţiei $i(v-r)-2(u+iv)q\zeta+i(v+r)q^2\zeta^2=0$).

3. Integrând (7) obţinem că funcţia extremală $w=f(\zeta)$ este dată implicit de ecuaţia:

$$\begin{cases} (1-2x\cos\theta+x^2\cos 2\theta)[2x(v\sin\theta-u\cos\theta)-x^2(v+r)\sin 2\theta]- \\ -2x\sin\theta(1+x\cos\theta)[v-r-2x(u\sin\theta+v\cos\theta)+x^2(v+r)\cos 2\theta]= \\ =(1-x^2)^2\operatorname{Re} f, \\ 2x\sin\theta(1+x\cos\theta)[2x(v\sin\theta-u\cos\theta)-x^2(v+r)\sin 2\theta]+ \\ +(1-2x\cos\theta+x^2\cos 2\theta)[v-r-2x(u\sin\theta+v\cos\theta)+x^2(v+r)\cos 2\theta]= \\ =\operatorname{Im}[-(x-a)(1-x^2)^2\ell], \\ \dfrac{5x^3-4ax^2-3x+2a}{(1-x^2)(x-a)}+\dfrac{2x(x-\cos\theta)}{1-2x\cos\theta+x^2}+\dfrac{N_1}{N_2}=\operatorname{Re}\dfrac{xm}{\ell}, \\ -\dfrac{\sin\theta}{1+2x\cos\theta+x^2}+\dfrac{N_3}{N_4}=\operatorname{Im}\dfrac{xm}{\ell} \end{cases} \qquad (8)$$

where:

$$N_1 = -2x(v-r)(u\sin\theta + v\cos\theta) + 2[2(u^2+v^2) + (v^2-r^2)\cos 2\theta]x^2 + \\ + 6(v+r)(u\sin\theta - v\cos\theta)x^3 + 2(v+r)^2 x^4;$$

$$N_2 = (v-r)^2 - 4x(v-r)(u\sin\theta + v\cos\theta) + 2[2(u^2+v^2) + (v^2-r^2)\cos 2\theta]x^2 + \\ + 4(v+r)(u\sin\theta - v\cos\theta)x^3 + (v+r)^2 x^4;$$

$$N_3 = 2x(v-r)(-v\sin\theta + u\cos\theta) + 2(v^2-r^2)x^2\sin 2\theta + 2(v+r)\times \\ \times(v\sin\theta + u\cos\theta)x^3 - 4(v+r)(v\sin\theta + u\cos\theta)x^3;$$

$$N_4 = (v-r)^2 - 4x(v-r)(u\sin\theta + v\cos\theta) + 2[2(u^2+v^2) + (v^2-r^2)\cos 2\theta]x^2 + \\ + 4(v+r)(u\sin\theta - v\cos\theta)x^3 + (v+r)^2 x^4.$$

By (7) we obtain:

$$w(\zeta_1) = \frac{f[f+(x-a)\ell]}{f+2(x-a)\ell} \qquad (9)$$

where $\zeta_1 = \rho\cdot\bar{q}$, $|\zeta_1|<1$, $\rho = \dfrac{u+iv-\sqrt{u^2-r^2+2uv\cdot i}}{i(v+r)}$ (ζ_1 is root of equation $i(v-r)-2(u+iv)q\zeta+i(v+r)q^2\zeta^2=0$).

3. By integrating equation (7) we obtain that the extremal function $w=f(\zeta)$ is given implicitly by the equation:

$$\left\{ \begin{aligned} &\frac{\rho\bar{q}k}{k-\rho^2}\,\frac{f+2(x^3-a^3)l}{f[f+(x^3-a^3)l]}\left(\frac{s_1+1}{s_1-1}\right)^{s_1}\frac{\sqrt{f^2[f+(x^3-a^3)l]}+\sqrt{f^2[f+(x-a)l]-f[f+2(x-a)l]w}}{\sqrt{f^2[f+(x^3-a^3)l]}-\sqrt{f^2[f+(x-a)l]-f[f+2(x-a)l]w}}\times \\ &\times\left(\frac{\sqrt{-f^2(x-a)l}+\sqrt{f^2[f+(x-a)l]-f[f+2(x-a)l]w}}{\sqrt{-f^2(x-a)l}-\sqrt{f^2[f+(x-a)l]-f[f+2(x-a)l]w}}\right)^{s_1} = \\ &=\left(\frac{\sqrt{k}+\rho}{\sqrt{k}-\rho}\right)^{\frac{1}{\sqrt{k}}}\left(\frac{\sigma\sqrt{k}+\rho}{\sigma\sqrt{k}-\rho}\right)^{s_2}\left(\frac{\tau\sqrt{k}+\rho}{\tau\sqrt{k}-\rho}\right)^{s_3}\left(\frac{1-y}{1+y}\right)^{\frac{1}{\sqrt{k}}}\frac{\rho+k\sqrt{y}}{\rho-k\sqrt{y}}\left(\frac{\sigma-y}{\sigma+y}\right)^{s_2}\left(\frac{\tau-y}{\tau+y}\right)^{s_3} \end{aligned} \right\} \qquad (10)$$

unde: $k = \frac{v-r}{v+r}$, $y = \sqrt{\frac{\zeta - \zeta\overline{q}}{\zeta - \frac{k}{\rho}\overline{q}}}$,

$$\sigma^2 = \frac{x\rho - \overline{q}\rho^2}{x\rho - \overline{q}k},\ \tau^2 = \frac{\rho - x\overline{q}\rho^2}{\rho - x\overline{q}k},$$

$$s_1 = \sqrt{\frac{-\ell(x-a)}{f + (x-a)\ell}},$$

$$s_2 = \frac{(x - \rho\overline{q})(x-q)}{\sigma\sqrt{k}(1-x^2)} \text{ şi}$$

$$s_3 = -\frac{(x\rho - q)(x-\overline{q})}{\tau\sqrt{k}(1-x^2)}.$$

Dacă facem $\zeta \to x$ în (7) obținem $s_1=s_2$. Folosind (9) şi (10) obținem:

$$f = \frac{\rho\overline{q}k}{k-\rho^2}(1-s_1^2)\left(\frac{s_1+1}{s_1-1}\right)^{s_1} \cdot \left(\frac{\sqrt{k}-\rho}{\sqrt{k}+\rho}\right)^{\frac{1}{\sqrt{k}}} \cdot \left(\frac{\sigma\sqrt{k}-\rho}{\sigma\sqrt{k}+\rho}\right)^{s_1} \cdot \left(\frac{z\sqrt{k}-\rho}{z\sqrt{k}+\rho}\right)^{s_3} \tag{11}$$

Valorile ρ, q, k, σ, z, s_1 şi s_2 ce apar în (10) depind de u, v şi r ; (8) şi (9) determină u, v şi r ca funcții de θ, şi θ este obținut din (9) şi $w'(0)=a_2$.

Cu f şi ℓ cunoscute, $\overline{x}$ este obținut din : $\overline{x}=\max\{x|\mathrm{Re}[3x^2f+(x^3-a^3)\ell]=0\}$.

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where: $k = \frac{v-r}{v+r}$, $y = \sqrt{\frac{\zeta - \zeta\overline{q}}{\zeta - \frac{k}{\rho}\overline{q}}}$,

$$\sigma^2 = \frac{x\rho - \overline{q}\rho^2}{x\rho - \overline{q}k},\ \tau^2 = \frac{\rho - x\overline{q}\rho^2}{\rho - x\overline{q}k},$$

$$s_1 = \sqrt{\frac{-\ell(x-a)}{f + (x-a)\ell}},$$

$$s_2 = \frac{(x - \rho\overline{q})(x-q)}{\sigma\sqrt{k}(1-x^2)} \text{ and}$$

$$s_3 = -\frac{(x\rho - q)(x-\overline{q})}{\tau\sqrt{k}(1-x^2)}.$$

If we put $\zeta \to x$ in (7) we obtain $s_1=s_2$. By using (9) and (10) we obtain:

$$f = \frac{\rho\overline{q}k}{k-\rho^2}(1-s_1^2)\left(\frac{s_1+1}{s_1-1}\right)^{s_1} \cdot \left(\frac{\sqrt{k}-\rho}{\sqrt{k}+\rho}\right)^{\frac{1}{\sqrt{k}}} \cdot \left(\frac{\sigma\sqrt{k}-\rho}{\sigma\sqrt{k}+\rho}\right)^{s_1} \cdot \left(\frac{z\sqrt{k}-\rho}{z\sqrt{k}+\rho}\right)^{s_3} \tag{11}$$

The value of ρ, q, k, σ, z, s_1 and s_2 which appear in (10) depend of u, v and r; (8) and (9) determine u, v and r as functions of θ, and θ is obtained from (9) and $w'(0)=a_2$.

Thus f and ℓ being known, $\overline{x}$ is obtained from the condition: $\overline{x}=\max\{x|\mathrm{Re}[3x^2f+(x^3-a^3)\ell]=0\}$.

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